THE MEASUREMENT AND INTERPRETATION OF SMALL STRAIN STIFFNESS OF SOILS

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Abstract: The measurement of soil stiffness at very small strains ($G_{\text{max}}$) has been carried out under both dynamic and continuous loading in the triaxial apparatus up to high stresses. A new system of LVDTs has been used to measure axial strain locally during continuous loading while dynamic stiffnesses were measured using bender elements. A good agreement was found using the two methods between the stiffnesses at 0.0001% strain for two different materials.

Bender elements measure the propagation time of shear waves through a soil sample so that $G_{\text{max}}$ can be determined. Bender elements were incorporated into high-pressure triaxial cells and were used to test very stiff soils. Theoretical studies and dynamic finite element analyses are presented, which have been carried out to develop more objective criteria for the determination of $G_{\text{max}}$.

The paper presents the results of bender element tests examining the variation of $G_{\text{max}}$ for sands, which is then related to the stiffnesses at larger strains determined under continuous loading using the new system of LVDTs. Three sands with very different geological origins were tested over a wide range of stresses allowing a general framework for stiffnesses to be established. The interpretation of the results is based on the correct normalisation of the data by which means unique relationships were derived for each soil. The framework demonstrates that the confining stress and volumetric state relative to the normal compression line are the principal controlling factors of stiffness for sands, as they would be for clays. However, the framework distinguishes that for sands the means of arriving at its initial volume-stress state are also important, in particular whether this is by overconsolidation or compaction.

INTRODUCTION

The research described in this paper presents in parts the research work on my PhD Thesis, which I accomplished at City University, London, UK, in 1997. The main topic of the research was the stiffness behaviour of soils and soft rocks at very small and small strains. The very small and small strain stiffness is thought to be particularly relevant for the prediction of soil behaviour at small strains and is required for the most non-linear soil models. The paper focuses on two topics: the bender element measurement technique and the framework for the stiffness behaviour of sands.

For the majority of soils yielding occurs at strains, which are very small i.e. less than 0.001% (Atkinson & Sallfors, 1991) and at subsequent small strains the stress-strain behaviour becomes highly non-linear, as shown in Figure 1. Very small strains are typically associated with the soil response to dynamic loading. It is now well established that small strain behaviour also plays an important role in soil response to static loading. Simpson (1992), Burland (1989) and others
showed that the strain level around engineering structures is in the range of small to moderate strains (up to 0.2%) emphasising the importance of evaluating the decay of stiffness with strain. The benchmark parameter of this characteristic stiffness-strain curve for any soil, which is not currently yielding is the very small strain shear modulus or $G_{\text{max}}$, which is sometimes also referred as $G_0$.

Fig. 1  Small strain stiffness of soils (after Atkinson & Sallfors, 1991)

Values of $G_{\text{max}}$ can be measured in the laboratory using resonant column test or other dynamic tests based on wave propagation methods. In this research $G_{\text{max}}$ was determined from the results of bender element tests, which measured the propagation time of a shear wave passing through the soil sample. Stiffnesses were also determined from the continuous loading tests in which the measurements of very small strains were carried out using local axial displacement transducers attached directly to the sample.

Previous research work on the small stiffness behaviour of sands indicated that $G_{\text{max}}$ varies with the state and history of the soil (Hardin & Richart, 1963; Hardin & Blanford, 19893). But majority of these tests were carried out at moderate stresses and that prevented researchers relating their findings with fundamental material parameters, which can only be identified at significantly higher pressures. In consequence, a general framework for the stiffness behaviour of sands has not been established. The research was designed to test sands at considerably higher pressures than before with an aim of relating the stiffnesses not only to the confining stress but also to the current volumetric state of the soil. Three reconstituted sands with different geological origins were tested with an aim to establish a new general framework for stiffness for sands.

1. MEASUREMENT OF SMALL STRAIN STIFFNESS IN THE LABORATORY

1.1 Introduction

The bender element technique was developed by Shirley and Hampton (1977) and was first applied to the triaxial apparatus by Shulteiss (1982) and Dyvik and Madshus (1985). It is a relatively simple method for deriving the elastic shear modulus of a soil at very small strains. The advantage of the bender element method when combined with the triaxial apparatus is that
allows any number of non-destructive measurements of $G_{\text{max}}$ during the course of a conventional stress path test.

The tests were carried out in a Bishop and Wesley (1975) type stress path cell, which is also schematically illustrated in Figure 2. This set-up is equipped with computer logging and control of the stresses and strains (Atkinson, Evans and Scott, 1985). The instrumentation used was comprised of two pressure transducers for the cell and pore pressures, an internal load cell for the deviatoric force, a volume gauge and an LVDT mounted externally to the cell to measure the axial strain. This apparatus had a sample size of 38mm diameter and a maximum capacity of 700kPa cell pressure. Tests at greater pressures were conducted in a high pressure stress path apparatus with a capacity of 5MPa (Taylor & Coop, 1990) the instrumentation of which was similar to that of the Bishop and Wesley cell, but which had a sample diameter of 50mm.
In some of the cells small water-submersible LVDTs were used for the local measurement of axial strains. The LVDTs were chosen in order to achieve measurements of stiffness at strain levels comparable to those caused by the dynamic loading of the bender element. A pair of RDP D5/200 LVDTs was used (Fig. 3) with a through bobbin bore and a right cable entry which allows free passage of the armature through the transducer at large strains.

1.1.1 Theoretical background to bender element method

The theoretical and numerical studies have been carried out to examine specific aspects of the bender element measurement technique in the triaxial apparatus. The bender element configuration in the triaxial apparatus, as shown in Figure 2, was used to measure the time of the propagation of a shear wave through the soil sample. Excited by the external voltage the transducer element moves and acts as a source releasing energy into the soil so that shear stresses propagate away through soil sample. The waves are captured by the receiver, which is aligned so as to detect the transverse motion which travels with the shear stresses. The travel time $T_a$ is measured as a time distance between the characteristic points on the screen of the oscilloscope, as shown in Fig. 4.
Characteristic points of measurements of the arrival time by bender elements

Point 1 corresponds to the start of the transmitter motion, which is in this case a single sine pulse. This represents the moment of energy transfer from the source to the soil. Point 1' corresponds to the start of the receiver motion and represents the moment of the energy transfer from the soil to the element. Assuming that the strains induced by the bender elements to the soil are elastic and knowing the current tip to tip distance $L_a$ between the elements the velocity of shear waves is calculated by:

$$v_s = \frac{L_a}{T_a}$$  \hspace{1cm} (1.1)

so that the very small strain (i.e. elastic) shear modulus $G_{\text{max}}$ is determined from:

$$G_{\text{max}} = \rho v_s^2$$  \hspace{1cm} (1.2)

where $\rho$ is the mass density of a dry sample or total mass density of a saturated sample.

1.1.2 **Practical consideration on the usability of the method**

The bender element method is based on the following simplifying assumptions, which underlay all wave propagation methods:

(i) the strains induced in the soil by the transmitter are very small, i.e. the soil response to the dynamic loading is elastic
(ii) the travel distance of the shear wave corresponds to the tip to tip distance between the elements

(iii) the body wave imposed by the transmitter to the soil is a plane shear wave, i.e. only transverse motion travels with the velocity of the shear wave

(iv) the soil sample acts as an infinite medium for the given source-receiver configuration, i.e. all the waves reflected from the boundaries of the sample arrive later at the receiver than the direct wave originating from the transmitter

There is a very little evidence to support directly assumption (i) since it is virtually impossible to measure the actual strain, which occurs at the contact between the soil and the element. In the course of this testing programme the stiffnesses of some samples were low but no accumulation of the strain was ever observed in the sample due to the dynamic loading. Taking into account that the dynamic loading was essentially cyclic this fact suggests that the soil developed recoverable, i.e. elastic strains only.

Viggiani (1992) established the validity of assumption (ii). Assumption (iii) is generally not valid. For the range of distances and frequencies typically used in the bender element test in the triaxial apparatus the waves generated by the point source represented by the transmitter cannot be considered plane but are spherical, so that it is not the case that only transverse motion travels with the velocity of the shear wave. Spherical front waves exhibit a far more complex propagation and polarisation pattern than plane waves and, theoretically, the absolute separation of the body wave into shear and compression components is not possible.

The most comprehensive analytical solution to the problem of the body wave propagation through the soil is given by Sanches-Salinero, Roeset & Stokoe (1986). The subject of the study was the propagation of a wave front originating from a point source through an infinite and isotropic elastic medium. The source point is subjected to motion of the shape of a sine pulse, which is transversely polarised relative to the position of the receiver. The shape of the resulting wave is defined by the variable \( \Gamma \) (Fig. 5) where \( v_p \) is the velocity of a compression wave, \( \omega \) is the angular velocity and \( d \) is the source receiver distance. This wave is far from being a simple transversely polarised shear wave propagating in a longitudinal direction as assumed by the bender element method.

\[
\Gamma_s = \frac{1}{d^3} e^{-\frac{i \omega d}{v_s}} + \left( \frac{1}{\omega d^3} - \frac{1}{v_p^3} \right) e^{-\frac{i \omega d}{v_s}} - \left( \frac{v_p^2}{v_s} \right)^2 \left( \frac{1}{\omega d^3} - \frac{1}{v_p^3} \right) e^{-\frac{i \omega d}{v_p}}
\]

\[
\Gamma_s = \Gamma_1 + \Gamma_2 - \Gamma_3
\]

Fig. 5 The function of the motion of a spherical wave broken into parts

It was found that variable \( \Gamma \) have three coupled components, corresponding to the three terms of the solution as a sum of three terms \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) which all represent transverse motion but which travel with different velocities. The first two (\( \Gamma_1 \) and \( \Gamma_2 \)) travel with the velocity of a shear wave, and the third (\( \Gamma_3 \)) travels with the velocity of a compression wave. For the three components the attenuation arising from geometric damping occurs at different rates, the second
and third terms (Γ2 and Γ3) attenuate one order of magnitude faster than term Γ1. Terms Γ2 and Γ3 are known as a near field waves since they are detected only near the source. Term Γ1 is a pure shear wave and is the only one detected in the far field as far from the source the near field waves attenuate to the point where they become negligible. Geometric damping therefore separates the near field coupled compression and shear waves from the far field, pure shear waves. Sanches-Salinero et al., (1986) expressed their results in terms of a ratio denoted here as \( R_d \). They recognized that this will control the shape of the receiver trace at the point of monitoring through the degree of attenuation due to geometric damping. The value of \( R_d \) is defined as the ratio between the source-receiver distance \( d \) and the wavelength of the source wave \( \lambda_w \):

\[
R_d = \frac{d}{\lambda_w} \quad \lambda_w = \frac{v_s}{f}
\]

so that:

\[
R_d = f \frac{d}{v_s}
\]

and \( f \) is the frequency of the source wave. According to the closed form solution this feature is clearly observed only for values of \( R_d \) higher than 2.0. For low values of \( R_d \) there is an initial downward deflection of the trace before the shear wave arrives, representing the near field effect given by the third component (Γ3). At high values of \( R_d \) the near field effect is almost absent. In conclusion, assumption (iii) is valid only in the absence of the near field waves. Being of predominantly one frequency, a single sine pulse is the most convenient choice for the shape of the transmitted wave.

Finally, assumption (iv) was not possible to check analytically since the complex boundary conditions of the bender element configuration in the triaxial apparatus prevent any closed form solution. Instead, finite element analyses were used to examine the independence of the method from the source-receiver configuration. A set of numerical analyses was carried out using the program package Solvia 90, which was chosen to perform analyses for which the geometry of the model is defined in three dimensional space (3D analyses) allowing the complex geometry of the prototype to be accurately modelled. The results of the analyses showed the identical pattern of response of the bender element as theoretically predicted, including the near field effect. The influence of the boundary conditions (assumption iv) is presented in Fig. 6, which shows the contours of the strain level on the cross-section of the sample in the plane of motion. The contours of strain are in the direct proportion to the contours of stresses since the material is modelled as elastic and for the same reason the values of strains are determined arbitrarily. The strain contours are given for three distinct time instants: immediately after the excitation, in the middle of the travel and at the moment of the arrival of the shear wave at the node which represents the tip of the receiver. It can be seen that in all three instants the wave front propagates along the sample axis and is parallel to the sample base. The shear disturbance becomes more widely spread along the sample with the time and the strain level declines due to geometric damping by about one order of magnitude by the time the wave reaches the monitoring point. There are no reflections from the sides of the mesh and the direct wave reaches the receiver first.
In conclusion, all the key assumptions related to the accuracy of the bender element method had been considered, and all of them have been found to be valuable under the controllable conditions. However, the method is still far from being a routine procedure, as it requires an educated judgment of a trained operator. Providing that the good coupling between element and the soil or rock is achieved (i.e. there is no overshooting) the bender element method should provide with reliable results.
2. STIFFNESS OF SANDS AT VERY SMALL AND SMALL STRAINS

2.1 Soils Tested

For consistency with the previous work on the mechanics of sands (Coop, 1990; Lee, 1991; Coop & Lee, 1993) the same three sands were used in this research. Dogs Bay sand, decomposed granite and Ham River sand had been originally chosen for their diversity of origins and characteristics. Dogs Bay sand, from the west of the Republic of Ireland, is a biogenic carbonate sand consisting largely of foraminifera and mollusc shells. The sand is poorly graded with a high calcium carbonate content (Houlsby et al., 1988). The soil was identified and gathered by Evans (1987) from a dune environment. The sand particles are relatively unbroken (Fig. 7a) and their open and angular nature gives rise to the soil's high voids ratios.

The Korean decomposed granite is a residual soil classified by a weathering grade V on the scale of the Geological Society (1990). It occurs naturally with a complex structure so in order to form a reconstituted soil the material was mechanically de-structured by breaking the inter-particle bonds. In the process, the particles greater than 5mm were discarded. Since the soil had not been transported it is well graded with angular and sub-angular particles (Fig. 7b). Decomposed granite is a product of chemical weathering so each particle is formed of an amalgam of different minerals. Quartz and feldspars were inherited from the parent rock while kaolin, mica and smectite are the results of the weathering process.

The Ham River sand is a quartz sand gathered from a Thames gravel quarry near Chertsey, England. Being typical of a river transported soil the sand is poorly graded with sub-angular to rounded particles (Fig. 7c).

Fig. 7 Scanning electron micrographs of typical soil particles: a) Dogs Bay sand b) decomposed granite c) Ham River sand
2.2 Stiffness at very small strains

2.2.1 Tests on Dogs Bay sand

The samples of Dogs Bay sand were prepared in as wide a range of initial densities as possible by using a variety of techniques (from dry compaction to wet pluviation). The volumetric paths during isotropic compression are given in the v:lnp' plane in Fig. 8a. Regardless of their initial densities all the samples tend towards a unique isotropic normal compression line (NCL). The approaching path is one of gradual yield, and being closer to the normal compression line, the loose samples reach the reference state at lower stresses than the dense ones. For example the loosest sample 9db reached a normally compressed state at 700kPa while the most dense sample 8db would have required around 3MPa to reach the normal compression line.

During unload-reload cycles the behaviour of all samples tested was very rigid with a well defined yield point on reloading when the preconsolidation pressure was reached. This typical recompression behaviour is the result of particle breakage which restarts at pressures higher than the preconsolidation pressure.

Fig. 8 Volumetric paths during isotropic compression for: a) Dogs Bay sand b) decomposed granite c) Ham River sand
Bender element measurements during the first loading of isotropic compression are given in Fig. 9a. Each bender element reading is shown on the graph as a data point. By direct comparison of the results from Fig. 8a and Fig. 9a it can be seen that initial density of the samples has a significant influence on the corresponding $G_{\text{max}}$ values. For a difference in specific volumes between the loosest sample 9db and the densest 8db of 17% a difference in $G_{\text{max}}$ of about 70% is observed. The decrease in initial density is followed consistently by a corresponding increase in the initial value of $G_{\text{max}}$ for all the samples tested.

It can be observed from Fig. 9a that on first loading the values of $G_{\text{max}}$ for all initial densities tend towards a unique line indicated on the figure which corresponds to the isotropic normal compression line in the $\nu$-$\ln p'$ plane. This result demonstrates that the state of normal compression is not only a reference state for the volumetric change of the soil but it is also a reference state for the very small strain stiffness. This unique line on Fig. 9a. appears to be straight in the log-log plot so it can be fitted by a power function of the form:

$$\frac{G_{\text{max}}}{p_i^{\prime \prime}} = 3100(p' / p_i)^{0.686}$$

$$\frac{G_{\text{max}}}{p_i^{\prime \prime}} = 760(p' / p_i)^{0.884}$$

$$\frac{G_{\text{max}}}{p_i^{\prime \prime}} = 3900(p' / p_i)^{0.593}$$

Fig. 9  Variation of $G_{\text{max}}$ with isotropic stress $p'$ for first loading a) Dogs Bay sand b) decomposed granite c) Ham River sand
\[
\frac{(G_{\text{max}})_{nc}}{p_r} = A\left(\frac{p'}{p_r}\right)^n
\]

where \((G_{\text{max}})_{nc}\) corresponds to the measured values of \(G_{\text{max}}\) at the states of normal compression and \(p_r\) is the reference pressure of 1kPa needed for the dimensional coherence of the equation. The same equation was used by Viggiani & Atkinson (1995) for the variation of \(G_{\text{max}}\) with \(p'\) for normally consolidated clays.

The stiffnesses measured during isotropic unload-reload loops in which all the samples were unloaded from the normal compression line are shown also in the \(\log G_{\text{max}}:\log p'\) plane in Fig. 10a. These data plot on curved lines above the reference \((G_{\text{max}})_{nc}\) line also presented in the figure. Each line corresponds to a swelling line of the particular sample (see Fig. 8a) so that the location of the line is determined by the value of the corresponding preconsolidation pressure. A small amount of hysteresis is observed in the volumetric variation during the same unload-reload cycle but the stiffnesses measured during reloading are indistinguishable from those measured during unloading.

Fig. 10 Variation of \(G_{\text{max}}\) with isotropic stress \(p'\) for unloading and reloading a) Dogs Bay sand b) decomposed granite c) Ham River sand
2.2.2 Tests on decomposed granite

The decomposed granite samples were prepared by wet or dry compaction after which carbon dioxide was circulated through the sample prior to flooding it so as to improve the subsequent saturation. In total five samples of decomposed granite were tested with a range of initial specific volumes varying between 1.490 and 1.605.

The gradual yield of the samples is seen in Fig. 8b as they tend towards a unique normal compression line. The gradient of normal compression line is almost four times lower than that for Dogs Bay sand resulting from a very low compressibility of this well graded soil. Similar features of unload-reload behaviour are seen as were observed for the Dogs Bay sand indicating a similarity in the mechanical behaviour caused by the particle breakage.

Bender element measurements of $G_{\text{max}}$ for the first loading of isotropic compression are given in Fig. 9b. The influence of initial density on the initial value of $G_{\text{max}}$ is also observed for this sand, the maximum difference in specific volumes of 8% resulting in about 50% difference in stiffness. As was observed previously for Dogs Bay sand, during the first loading of the samples of decomposed granite the values of $G_{\text{max}}$ tend towards a unique $(G_{\text{max}})_{nc}$ line regardless of the samples' initial densities. This line is again straight in the log$G_{\text{max}}$:log$p'$ plane so that the material parameters (see Equation 2.1) can be identified as $A=760$ and $n=0.884$. By direct comparison of Fig. 8b and Fig. 9b it can be observed that samples 1dg, 2dg and 4dg had not reached the normal compression line. Correspondingly the $(G_{\text{max}})_{nc}$ line had not been reached, but the tendency towards it is clear, as indicated by arrows in Fig. 9b.

Unload-reload stiffness data for the samples, which were unloaded from the normal compression line are given in Fig.10b. Here again no hysteresis is observed in terms of stiffness during the cycle and the locations of the lines are determined by the corresponding preconsolidation pressures. In contrast to the curved lines observed for Dogs Bay sand, the stiffness lines in this case appear straight in the log:log plot. However, the lines are in both cases parallel thus indicating that there is a relationship between the volumetric states and the corresponding values of $G_{\text{max}}$.

2.2.3 Tests on Ham River sand

The samples of Ham River sand were prepared by wet compaction, which was considered a consistent means of producing homogeneous samples with a variety of initial densities. A total of four samples was tested.

Volumetric paths in the v:lnp' plane are presented in Fig. 8c The tests on the loose sample 1hr and the dense sample 2hr were carried out in the high pressure apparatus in which the maximum capacity allowed during isotropic compression was only 5MPa. This pressure caused only a small change in volume of about 1.5%. In order to reach the reference state of normal compression a substantially higher pressure was needed and the two loose samples 4hr and 5hr were tested in the high pressure apparatus with a capacity of 70MPa. The gradual yield of the samples increased at around 10MPa which is a yield pressure one logarithmic cycle higher than observed for the two sands previously tested. However in the same manner as observed for the other two soils the samples reached a unique normal compression line, the position of which was also identified by Coop & Lee (1993). The gradient of the normal compression line of Ham River sand is about half that of Dogs Bay sand and about twice that of the decomposed granite.
The variation of $G_{\text{max}}$ on first loading of isotropic compression is given in Fig. 9c. As was observed for the other two materials, the initial density controls the initial value of $G_{\text{max}}$. This time the effect was much smaller so that the maximum difference in specific volumes of around 7% gave a difference in $G_{\text{max}}$ of about 25% which is about a third of the effect observed for Dogs Bay sand and about a half that observed for decomposed granite. This feature of Ham River sand that the volumetric state does not affect greatly $G_{\text{max}}$ is in contrast with the behaviour of the two other sands tested. It will be shown later that this characteristic behaviour of the silica sand persists at all stress levels.

As can be seen from the stiffness data in Fig. 9c the curvature of the approaching paths to the unique $(G_{\text{max}})_{\text{nc}}$ line for the samples with different densities is very small. A close inspection is needed to distinguish the straight portion of the line which corresponds to the reference state from the curved approaching paths. This again indicates that the mean effective stress is the dominant parameter which controls $G_{\text{max}}$ for this sand. The parameters which characterise the $(G_{\text{max}})_{\text{nc}}$ line of Ham River sand have been identified as $A=3900$ and $n=0.593$, as indicated in the figure.

The stiffnesses measured during isotropic unload-reload loops are shown in Fig. 10c. Again these data lie on straight lines above the $(G_{\text{max}})_{\text{nc}}$ line and each line corresponds to a swelling line in the $v:\ln p'$ space so that both are controlled by the corresponding preconsolidation pressure. A close inspection shows that the stiffness lines remain parallel at all pressures thus indicating again that there is a relationship between the volumetric state and the stiffness, even though the effect is small.

2.3 Framework for the behaviour of sands at very small strains

Three sands with different mineralogies and different geological origins showed a remarkable similarity in their behaviour. The question arises as to whether there is a unique framework which would be applicable to the stiffnesses of all three sands and, very likely, for reconstituted sands in general. Here, an attempt is made to develop such a framework by introducing the influence of volumetric state on stiffness using a similar approach to that previously found to apply to reconstituted clays by Viggiani & Atkinson (1995).

During the first loading of each sand it was observed that the values of $G_{\text{max}}$ for all samples at all initial densities eventually reach a unique line which corresponds to the isotropic normal compression line in the volumetric plane. This unique line of very small strain stiffness of a normally consolidated sand (i.e. the $(G_{\text{max}})_{\text{nc}}$ line) can be characterised by Equation 2.1 using the gradient of the line $n$ and its intercept $A$ in the $\log G_{\text{max}}:\log p'$ plot. For the three sands tested it was observed that on first loading the stiffnesses on the approach paths to the $(G_{\text{max}})_{\text{nc}}$ line depend on the samples' initial densities. As already mentioned, the gradients of the curved approaching paths are approximately in the range of 0.5 to 0.7 which is close to the values given in the literature.

The stiffnesses at both overconsolidated and compacted states were found, for all three soils, to be greater than at normally consolidated states for the same mean effective stress. An attempt has then been made to examine whether there is a unique relationship between the volumetric state and stiffness, taking into account both ways of how a given state may be reached, i.e. by overconsolidation or by compaction and first loading. In this context the term overconsolidated will be reserved for those samples, which are currently at stresses which are below the
preconsolidation pressure that they have experienced in a state of isotropic normal compression. The term compacted will be used to define those samples which are undergoing first loading prior to reaching the isotropic normal compression line whatever their initial density.

The normalisation procedure for defining the volumetric state of the samples was chosen to be one, which would not depend on the means of how the volumetric state is reached. The current volumetric state has been determined by normalising the current mean effective stress $p'$ with respect to the equivalent pressure $p'_e$ taken at the same specific volume on the normal compression line (see Fig. 11) where:

$$p'_e = \exp\left(\frac{N - v}{\lambda}\right)$$  \hspace{1cm} (2.2)

and where $\lambda$ is the gradient of the normal compression line and $N$ its projected intercept at a $p'$ of 1kPa. The use of the state variable $p'/p'_e$ is analogous to using overconsolidation ratio for clays.

![Fig. 11 State variables for sands](image)

The values of $G_{\text{max}}$ for the overconsolidated and compacted samples have also been normalised relative to the reference state in terms of stiffness, i.e. the ($G_{\text{max}}$)$_{hc}$ line representing the values of $G_{\text{max}}$ for normally consolidated states at the current value of mean effective stress. The values of ($G_{\text{max}}$)$_{hc}$ have been calculated from Equation 2.1 so that:

$$G_{\text{max}} = p_r \left(\frac{p'}{p_r}\right)^n$$  \hspace{1cm} (2.3)
In the same way that for $p'_e$ a projection to the normal compression line at the current volume is made, for $G_{\text{max}}$ the projection is made to the $(G_{\text{max}})_{nc}$ line at the current value of mean effective stress $p'$. By plotting normalised stiffnesses, $G_{\text{max}}/(G_{\text{max}})_{nc}$, versus normalised volumetric states, $p'/p'_e$, all of the stiffnesses of samples lying on the normal compression line plot as a single point at 1:1. This is shown in Fig. 12 where normalised plots are given in the form of semi-logarithmic graphs for the three sands. The same pattern of behaviour can be recognised for each material. All the compacted samples at all volumes define one unique relationship for normalised stiffness with volumetric state. Similarly, all the overconsolidated samples define different but also unique lines for all samples at all volumes. It is observed that for both overconsolidated and compacted samples there is a general increase of normalised stiffness with decreasing state variable $p'/p'_e$, i.e. with increasing distance from the normal compression line for compacted samples or with increasing overconsolidation for overconsolidated samples.

![Graphs showing variation of $G_{\text{max}}$ with normalised volumetric state](image1)

Fig. 12 Variation of $G_{\text{max}}$ with normalised volumetric state
The basis of the framework is the uniqueness of the lines, which are the graphical representations of unique mathematical descriptions of the relationships which hold between $G_{\text{max}}$ and stress-volume state for the three sands. That is, the value of $n$ (Equation 2.1) does not vary with volumetric state (i.e. $p'/p'_{c}$) so that for a given state the $\log G_{\text{max}}: \log p'$ relationship for either compacted or overconsolidated samples would plot as a straight line parallel to and above the $(G_{\text{max}})_{nc}$ lines.

There is a substantial difference between the influence that volumetric state has on stiffness for the various sands tested. While the normalised stiffness of overconsolidated samples of Dogs Bay sand is nearly linear when plotted against normalised state with semi-logarithmic scales, for the decomposed granite it is distinctively curved (Fig. 10b). However the magnitude of the increase of stiffness with increasing distance from the normal compression line is rather similar, which may arise from the fact that both sands have angular particles prone to crushing.

The influence of the state on stiffness for Ham River sand is the least pronounced of the three sands (Fig. 12c). The maximum measured difference in stiffness between the most heavily overconsolidated samples (i.e. with the lowest $p'/p'_{c}$ values) and the corresponding compacted samples is only about 25% which is about one third that observed for the two other soils. It should also be noted that this effect is seen at an order of magnitude lower value of the state variable $p'/p'_{c}$ than for the two other sands, thus emphasising the difference in the behaviour. This may be explained by the fact that silica sand has rounded and solid particles, unlike the carbonate sand and decomposed granite, which is perhaps the cause of a less pronounced tendency in this material to develop new contacts between particles during compression.

As can be observed by the direct comparison of the graphs in Fig. 10 the size of the normalised stiffness domain (i.e. the area between the two lines) varies considerably between the three sands. The size of the domain is most prominent for the carbonate sand which is the most prone to particle breakage and could, therefore, result in there being a greater effect of crushing on the nature of the particle contacts. In the case of the decomposed granite the stiffness domain is small perhaps indicating that for this well graded soil made of angular particles, the breakage adds little to the nature of the particle contacts and to the already high potential for developing new contacts between particles during compression. Ham River sand has the least prominent size of the stiffness domain perhaps showing that the nature of contacts between solid and rounded particles was less affected by particle breakage.

### 3. THE STIFFNESS OF SANDS UNDER CONTINUOUS LOADING

#### 3.1 Tests on Dogs Bay sand

The shearing tests under continuous loading on the 38mm diameter samples were conducted in a standard Bishop & Wesley triaxial cell. The cell was equipped with both bender elements and a pair of miniature LVDTs so that a comparison between stiffnesses obtained using dynamic and continuous loading could be made for the same sample, as shown in Fig. 2. The system of miniature LVDTs allows the stiffnesses to be established at strains down to about 0.0001%, which was regarded as being comparable to the strains imposed in the soil by bender elements.

The test programme for each sample consisted of successive stages of isotropic compression and undrained shearing. Between the stages of isotropic compression and swelling, undrained shearing probes were carried out using a constant rate of strain of about 0.1% per hour. The test
programme which consisted of 15 probes during three multi-stage tests. A typical stress-strain curve for a probe is given in Fig. 13 where an average strain for the two transducers is plotted against deviator stress at two different scales. For each transducer about 500 data points were logged not of all which are shown in the figure. The difference between the two transducer readings was typically less than 5% of the current reading and an average value was used for the calculation of the tangent stiffness. Some typical stiffness-strain curves obtained at different initial mean effective stresses, $p'_i$ are given in Fig. 14a. The tangent stiffness has been defined as:

$$G = \frac{1}{3} \frac{\delta q}{\delta \epsilon_a}$$

3.1

Fig. 13 Stress-strain curve for an undrained shearing probe at $p' = 63$ kPa for Dogs Bay sand
and has been calculated using a linear regression typically through 15 points of the stress-strain curve. Fewer points were used (typically 7–9) at the beginning of the curve where the data were more scarce. The resulting tangent stiffness then corresponded to the middle point of the strain interval. At very small strains (i.e. below 0.0001%) the random noise dominates, giving a wide variation in the stiffnesses calculated which does not represent the soil behaviour. The values of stiffness below 0.0001% have therefore been omitted from the graphs.

In many cases stick-slip behaviour also restricted the minimum strain at which the stiffness could be defined. Out of total of 15 shearing probes carried out, only two probes gave the possibility of calculating the stiffness at 0.0001% strain.

The stick-slip effect was worse at higher pressures as the flexibility of the loading system increased relative to the stiffness of the sample and was particularly bad for samples which had been unloaded prior to shearing. Typical stiffness-strain curves for overconsolidated samples are given in Fig. 14b where because of the stick-slip behaviour the stiffness data have generally being omitted at strains lower than 0.001% except for the probe sh8 where $p_1'$ was relatively low (150kPa). These samples were not truly overconsolidated since the capacity of the cell was not sufficient to bring the samples to the reference state of normal compression.

3.2 Comparison between $G_{\text{max}}$ measured using dynamic and continuous loading

As indicated in the previous section, the instrumentation of the apparatus allowed a direct comparison to be made between stiffnesses obtained during dynamic and continuous loading of a single sample. These comparisons are also presented in Fig. 14a in which the bender element measurements are given along with the stiffness-strain curves for the samples during first loading (i.e compacted samples). A good agreement is seen between the dynamic and continuous loading stiffnesses at a strain of 0.0001%. In contrast, for overconsolidated-compacted samples the first reliable stiffnesses from the continuous loading measurements were obtained at about 0.001% so that direct comparisons were not possible (Fig. 14b).

![Fig. 14 Variation of G with strain for a set of undrained triaxial compression tests](image-url)
In order to overcome the problem that only a few data are available at the smallest strain of 0.0001% an attempt has been made to find an objective way of comparing the stiffness-strain curves obtained at different stress levels. This has been done by normalising the current stiffness with respect to \( G_{\text{max}} \) so that curves from different stress levels can be superimposed.

Viggiani & Atkinson (1995) showed that at small strains the stiffnesses at different strain levels for reconstituted Speswhite kaolin in a normally consolidated state depends on the mean effective stress. They used an equation of the form:

\[
\left( \frac{G}{p_r} \right)_{nc} = A_s \left( \frac{p'}{p_r} \right)^{n_s}
\]

where \( nc \) denotes the normally consolidated state. The parameters \( A_s \) and \( n_s \) are associated with a particular strain level and were determined from a continuous loading path in which \( p' \) was held constant. Here, the stiffnesses obtained during the undrained shearing probes at larger strain levels for the compacted samples of Dogs Bay sand are compared with the \( G_{\text{max}} \) values obtained from the bender element tests. This is shown in Fig. 15 in which the variation of undrained shear stiffness \( G \) is presented at different strain levels along with the \( (G_{\text{max}})_{nc} \) line. The data from the two continuous loading tests in which the stiffness was obtained at 0.0001% strain can be seen to plot on the \( (G_{\text{max}})_{nc} \) line along with the bender element data.

![Fig. 15 Variation of G with stress and strain for first loading of Dogs Bay sand](image)

A set of lines sub-parallel to the \( (G_{\text{max}})_{nc} \) line is seen for the larger strains in the same figure. These lines converge towards higher stresses indicating that the stiffness reduces at a higher rate with increasing stress level and which can be also seen from the individual stiffness-strain curves given in Fig. 14. Equation 3.2 is also used here to fit the stiffness lines at any particular strain level, as indicated in the figure.

The convergence of the stiffnesses with increasing \( p' \) is described graphically in Fig. 16, in which the values of the parameters \( A_s \) and \( n_s \) are plotted against the corresponding strain level.
Fig. 16 Variation of the parameters $A$ and $n$ with strain level for Dogs Bay sand

Fig. 17 Variation of normalised tangent stiffness with normalised volumetric state for compacted-overconsolidated samples of Dos Bay sand
It can be seen that the value of the parameter $A_s$ decreases towards zero while the power coefficient $n_s$ increases with the strain level. It was not possible to examine whether the power coefficient would reach a value of 1 at larger strains since the probes could not be continued to such high strains.

The tangent stiffnesses of the overconsolidated-compacted samples obtained for different strain levels are presented in Fig. 17 within the framework presented. Here a logarithmic $G/(G_{\text{max}})_{nc}$ scale has been used for convenience to plot the larger strain data. The $G_{\text{max}}$ line for truly overconsolidated states has been re-plotted here and lies above the $G_{\text{max}}$ values. As previously discussed, the bender element measurements were taken during unloading but since the samples never reached the state of normal compression their normalised stiffnesses plot within the stiffness zone defined for overconsolidated-compacted samples. The normalised stiffness data for different strain levels show a clear pattern of unique lines in which the stiffness increases with the distance from the normal compression line (i.e. as $p'/p'_c$ reduces) with approximately the same gradient for all strain levels. This result demonstrates that the stiffnesses at small strains, as was shown for the stiffnesses at very small strains, can be described within this framework, again illustrating the influence of volumetric state on stiffness.

CONCLUSIONS

Bender element measurement technique, together with the use of local instrumentation (LVDTs) in the triaxial cell, was proved to be an efficient experimental set up to examine the small strain behaviour of sands. Theoretical, numerical and practical considerations on the usability of the bender element technique demonstrated that the method, when used by an educated operator, should provide with reliable results.

Tests on three sands with different mineralogies and geological origins have been carried out in order to examine the influence of volumetric state on stiffness at very small strains. High pressures were used to bring the soil samples of various densities to the state of isotropic normal compression. This state was used as the reference state for the stiffness at very small strains, the latter being measured using bender elements. The stiffnesses at other states have been compared with the stiffnesses at this reference state and, by means of correct normalisation of the data, a new framework was proposed, which emphasises not only the influence of the confining stress on stiffness but also that of volumetric state.

A particular feature of the framework is the distinction between truly overconsolidated sands and those, which have only undergone first loading, as it was found that they have significantly different stiffnesses at the same volumetric state. Between these benchmarks a zone of possible stiffnesses has then been identified in which stiffnesses for all combinations of compaction and overconsolidation would be found. Since the initial compaction of a soil deposit would be controlled by the means of deposition, the framework highlights the influence that the geological history of the soil and its subsequent loading history would have on stiffness.

The framework is based on the behaviour of three sands originally chosen for the diversity of their properties, so it is likely that it is generally applicable. In order to obtain a complete understanding it was necessary to use high pressure testing, but it should be emphasised that the framework is applicable for all stress levels. The understanding upon which the framework is
based gives an explanation why the influence of volumetric state on stiffness has, so far, received little attention.

A new system of local axial measurement of strain has been used in the triaxial apparatus, which was sufficiently accurate to allow comparison of continuous loading and dynamic stiffnesses. It was found that the bender element stiffnesses corresponded to the tangent stiffnesses measured under continuous loading at about 0.0001% strain and no evidence of an elastic plateau was identified for samples undergoing isotropic first loading. The system was also used to examine the stiffness behaviour of sand on first loading in the region of small strains. The stiffnesses obtained at larger strains during continuous undrained shearing probes were compared with those obtained at very small strains. A set of sub-parallel strain contours has been identified for which the stiffnesses converge towards higher stresses in the same manner as was previously observed for clays. The tangent stiffnesses of overconsolidated-compacted samples have also been presented within the new framework, highlighting the influence that volumetric state has on stiffness not only for very small strains but also in the region of small strains.

REFERENCES