Dilatancy in general Cambridge-type models

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Cambridge-type models for soil (Cam-clay, Modified Cam-clay and NorSand) are idealised soil models based on a few simple postulates. This gives the models predictive power, and they offer interesting insight into soil behaviour. But the symmetry of the triaxial conditions for which the models were derived leaves an additional degree of freedom when generalising to arbitrary 3-D stress and strain states. The additional freedom creates unresolved inconsistencies in the plastic strain rates—usually dilatancy from the plastic potential not matching that implied by the work dissipation postulate (flow rule) under general 3-D stress states. It is shown that Cambridge-type models may be consistently generalised for arbitrary strain paths by adopting conjugate invariants and by requiring primacy of the work dissipation postulate. The unresolved freedom for the strain rates is handled by interpolation between the limit conditions (triaxial compression and extension) that are fully defined because of symmetry. The approach is illustrated in the context of NorSand, using calibration under triaxial compression for published tests on Brasted sand to predict behaviour in the practically important case of plane strain. Excellent predictions are obtained over a range of sand densities from loose to dense.

KEYWORDS: constitutive relations; plasticity; numerical modelling and analysis

INTRODUCTION

Most constitutive models for soil are based on plasticity, which is in itself a macro-scale abstraction of the underlying micromechanical reality of grain realignments and movements. Plastic models for soil cover a philosophical range from descriptive to idealised. Descriptive models are intrinsically curve fitting, and are anchored to test data; they can be very suitable for stress analysis if the stress paths are similar in the problem to the test conditions. However, accuracy of descriptive models in representing a particular situation is offset by an absence of insight into the underlying physical processes. Idealised models start from postulated mechanisms, from which behaviours are then derived. Idealised models trade accuracy in a particular situation for a consistent, simple (and known) physics.

An interesting and well known group of idealised models are those developed by the Cambridge School (Roscoe et al., 1963; Roscoe & Burland, 1968; Schofield & Wroth, 1968; Jefferies, 1993, 1997). These models are Granta-gravel, Cam-clay, Modified Cam-clay and NorSand. We shall refer to them collectively as Cambridge-type (CT) models, although they are sometimes also referred to as critical state models. Key features of CT models include:

(a) stress invariants used to express behaviour in terms of the ratio of deviatoric to mean stress
(b) idealised mechanisms for plastic work dissipation leading to the stress-dilatancy rule
(c) yield surfaces derived from the stress-dilatancy rule assuming normality
(d) use of the critical-state locus to relate yield surface size to void ratio.

CT models are widely taught and form the basis of several texts, and variants of Modified Cam-clay are often one of the standard options in finite element codes for geomechanics.

The extension of CT models from the triaxial conditions used in their derivation to general stress states (for finite element analysis or the like) requires more than the common approach of replacing the triaxial invariants with 3-D equivalents. There are two basic difficulties. First, CT yield surfaces are predicated on a plastic work dissipation postulate, but this postulate is not work conjugate when expressed in the usual invariants of finite element analysis (\( \sigma_{ii}, \sigma_{ii}, \theta, \dot{e}_t, \dot{e}_p \)). This might seem like a detail, but it strikes to the heart of these idealised models. In CT models, the idealised work dissipation postulate leads to the flow rule, and the flow rule is then integrated assuming normality (following Drucker 1951, 1959) to get the yield surface. If the flow rule is inconsistent then the entire validity of the model becomes

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questionable. Second, CT models invoke normality, but normality in the $\pi$-plane (for general stress states represented by the Lode angle, $\theta$) violates the stress dilatancy implied by the work dissipation postulate unless the critical stress ratio $M \neq f(\theta)$. But self-consistency with the idealisation of a particulate material having no tensile strength requires that $M$ be a function of $\theta$.

Of course, it might be suggested that these difficulties are a consequence of CT models’ being based on normality, and that the proper approach is to use a non-associated formulation. The counter argument is simple. CT models have appealing applied mechanics, and fewer material parameters (which are also density independent) than equivalent non-associated models. CT models can also replicate the triaxial test well, even for dilatant sands (for example NorSand). On the basis of Occam’s razor, CT models should be preferred over non-associated formulations. Further, as will be shown in this paper, CT models can readily be made self-consistent for arbitrary stress combinations. The issue is care in generalising CT models, not a fundamental flaw.

The paper is in four parts. First we show how the postulated work dissipation of CT models may be consistently expressed in general stress and strain space, as that gives the basic framework. Second, an appropriate $M(\theta)$ is determined for this framework, based on published test data, as this affects the way in which CT models generalise. Third, a first-order approach to obtaining consistent dilatancy in the $\pi$-plane is given, and the need for this is demonstrated. Finally, the suggested approach is implemented in the NorSand model and validated against published test data.

Work conjugate stress and strain measures

CT models start from a proposition as to the way in which plastic work is dissipated, which leads to the derivation of the yield surface. Consistency issues arise in extending these models from triaxial to arbitrary stress conditions to ensure that the original idealisations are not compromised. This is most easily seen by example. In presenting this example the flow rule and yield surface of Granta-gravel/Cam-clay/NorSand will be used; those of Modified Cam-clay are slightly different, but the difference is only a detail and does not affect the argument.

Under triaxial conditions, for a mean effective stress $p$, a deviator stress $q$ and the corresponding plastic strain rates $\dot{\lambda}$, the plastic work rate of stresses, per unit volume of the element of soil, during a strain increment is

$$ W^p = q \dot{\lambda}^p + \dot{\lambda}_p^p $$

(for triaxial conditions) (1)

CT models dissipate external work with plastic shear at a rate given by $W^p = M \dot{\lambda}^p$. Equating the external work rate to the internal plastic dissipation gives the fundamental stress—dilatancy or plastic flow rule:

$$ D^p = M = \eta $$

in which, for triaxial tests, the stress ratio $\eta = q/p$ and the dilatancy $D^p = \psi^p/\psi^p$. Assuming normality, equation (2) is transformed to a separable and integrable equation giving the yield surface ($F = 0$):

$$ F = \eta - M \left[ 1 - \ln \left( \frac{p}{\eta} \right) \right] $$

(3)

The yield surface size is scaled by the stress $\eta$, which is controlled by the hardening law. In Granta-gravel, $\eta$ is identified with the mean effective stress at the critical state for the current void ratio; in Cam-clay, $\eta$ is identified with the mean effective stress at the critical state for the current void ratio after allowance for elastic volumetric changes; and in NorSand, $\eta$ is simply a hardening parameter that evolves with plastic shear strain at a rate that depends on the state parameter $\psi$ (which is defined in Fig. 1).

The above ideas are generalised from triaxial conditions to arbitrary stress combinations by identifying the original triaxial strain and stress measures with invariants. It is usual to adopt the stress invariants ($\sigma_0, \sigma_m, \theta$) first suggested by Lode (1926) and brought to familiarity in the English-speaking world by Nayak & Zienkiewicz (1972). These stress invariants are ubiquitous in modern numerical approaches to modelling soils (e.g. Smith & Griffiths, 1988). Substituting $\sigma_0 = q$ and $\sigma_m = \eta$ simply changes variable in equation (3), as the invariants reduce to the triaxial variables under triaxial conditions.

The usual approach to strain increments in a generalised CT model follows Zienkiewicz & Naylor (1972). Associated flow (normality) is assumed, and the strain rates are obtained from differentiating the yield surface:

$$ \dot{\psi}^p = \alpha \frac{\partial F}{\partial \psi} $$

(4)

where $\alpha$ is a coefficient of proportionality. There are two problems associated with generalising critical-state models in this way.

Fig. 1. Definition of state parameter, $\psi$, and overconsolidation ratio, $R$
First, because yield surface hardening most generally depends on plastic shear strain (volumetric hardening cannot be used to model soils showing S-shaped undrained stress paths or equivalent), what is the appropriate shear strain invariant to be associated with the mobilised dilatancy (equation (2))? It provides a relation between the two uncertain. The model is inconsistent, and the validity of the invariants is not work conjugate, then generalising equation (2) in terms of these invariants implies a plastic work dissipation that is entirely different from the original postulate. The first step in resolving these contradictions is to introduce work conjugate invariants. There is a free choice between either stress or strain invariants, so adopt \( \sigma_m, \sigma_t \) and \( \theta \), and use for hardening? Zienkiewicz & Naylor (1972) suggested that the appropriate strain invariants were \( \varepsilon_t, \varepsilon_v \). However, these strain invariants are not work conjugate with the adopted stress invariants other than under triaxial conditions. Specifically:
\[
\overline{\sigma}_m \varepsilon_t^m + \overline{\sigma}_m \varepsilon_v^m \neq \overline{\sigma}_t \varepsilon_t^m + \overline{\sigma}_t \varepsilon_v^m + \overline{\sigma}_v \varepsilon_v^m \quad \text{if} \quad \theta \neq 30, -30°
\]  
(5)
This leads to questionable yield surface validity since, if the invariants are not work conjugate, then generalising equation (2) to include (4) in itself specifies a dilatancy. The two approaches provide the same result only if \( \partial M/\partial \theta = 0 \), requiring constant \( M \) in the \( \pi \)-plane. But constant \( M \) implies a soil that has tensile strength (Bishop, 1966), contradicting a starting postulate and strongly at variance with test data.

On rearranging, the appropriate work conjugate shear strain measure is obtained:
\[
\chi_q = \frac{s_1 \varepsilon_1^p + s_2 \varepsilon_2^p + s_3 \varepsilon_3^p}{\overline{\sigma}_m} 
\]  
(7a)
where the principal deviatoric stress \( s_j = (2\overline{\sigma}_j - \overline{\sigma}_1 - \overline{\sigma}_2)/3 \) and others by cyclic rotation. On substituting the principal stresses, equation (7a) may be written as
\[
\chi_q = \frac{1}{3} \left[ (\sin \theta + \sqrt{3} \cos \theta) \varepsilon_1^p - 2 \sin \theta \varepsilon_2^p + (\sin \theta - \sqrt{3} \cos \theta) \varepsilon_3^p \right] 
\]  
(7b)
Because equation (7b) is linear it allows the usual elastic-plastic decomposition of strain. Further, \( \chi_q \) reduces to the triaxial variable \( \chi^p \) under triaxial conditions. Although the strain measure \( \chi_q \) is unfamiliar, it was first proposed nearly two decades ago by Resende & Martin (1985) in the very similar context of generalising Drucker–Prager cap models. This strain measure, together with the familiar volumetric strain \( \varepsilon_v \), ensures work conjugacy with \( \overline{\sigma}_q, \overline{\sigma}_m \) and \( \theta \), and correspondingly CT models may be generalised using \( \chi_q, \overline{\sigma}_q, \overline{\sigma}_m \). Provided that dilatancy is defined as
\[
D^p = \frac{\varepsilon_v^p}{\chi_q^p} 
\]  
(8)
Most importantly, equation (7b) and thus equation (8) is not a function of \( \eta \), so that the yield surface (equation (3)) obtained by integration of the stress dilatancy rule (equation (2)) under normality remains valid.

Of course, the assumption of coaxiality of strain increments and stress means that equation (7b) may need further refinement when dealing with the effects of principal stress rotation and the way in which they may be captured within a CT model. But consistency in the \( \pi \)-plane and over-specification of dilatancy can now be addressed for fixed principal stress directions. The starting point is the function for \( M \).

CRITICAL STRESS RATIO, \( M \)

The relationship between stresses at the critical void ratio has been investigated for triaxial compression over a wide range of stress (e.g. Vaid & Sasitharan, 1992). The data are largely for sands as this is experimentally convenient, but this is no restriction as CT models treat all soils as cohesionless. It is uncontroversial to take
\[
\overline{\sigma}_q = M \overline{\sigma}_m 
\]  
(9)
where equation (9) is independent of the critical void ratio. Following Bishop (1972), data from several tests are plotted in the stress–dilatancy form \( \eta_{\max} \) against \( D_{\min} \). \( M \) corresponds to \( \eta_{\max} \) at \( D_{\min} = 0 \), and Fig. 2 illustrates this for triaxial data on Erksak sand, giving \( M_{\max} = 1.26 \) in triaxial compression (the data are from Vaid & Sasitharan, 1992).

The lack of controversy over equation (9) does not extend to the effect of Lode angle on \( M \). Taking \( M \) as a material constant represents soil behaviour very poorly (Bishop, 1966). For example, Fig. 2 also shows Vaid & Sasitharan’s (1992) triaxial extension data on Erksak sand. Although there is a smaller range of dilation to define the trend, a trend is nevertheless evident and indicates \( M_{\max} = 0.82 \) in triaxial extension.

The effect of intermediate principal stress on the failure criteria of sand was actively researched throughout the 1960s and early 1970s (e.g. Cornforth, 1964; Bishop, 1966; Green & Bishop, 1969; Green, 1972; Reades, 1972; Lade & Duncan, 1975). This interest covered a wide range of sand densities, but was directed at peak strength with substantial dilatancy. However, for CT models, the zero dilation rate critical friction is of primary interest as this is the plastic work dissipation mechanism. The dilatant strength component merely transfers work between the principal directions.

Available data on \( M \) as a function of Lode angle are sparse, and the results obtained by Cornforth (1961, 1964) on Brasted sand are an important contribution that often
underlies more recent views of sand behaviour (e.g. that of Bolton, 1986). Although these Brasted sand data are some 40 years old, subsequent testing using more complex equipment (e.g. Reades, 1972) produced the same trends. Fig. 3 shows Cornforth’s data on Brasted sand as η_{max} against D_{min}. A linear trend fits the data, giving M_{tc} = 1.27 in triaxial compression and M_{tc} = 0.81 in triaxial extension. In plane strain, the best-fit trend line is still linear, and suggests that M_{ps} = 1.08.

However, plotting the plane strain data in Fig. 3 in comparison with those from triaxial testing makes the tacit assumption that plane strain is a unique stress condition, as the figure compares two triaxial conditions (for which the stress combination is unique) with that of plane strain. This assumption is incorrect. Fig. 4 shows the Lode angle at peak strength against dilation rate in the plane strain tests on Brasted sand. This variation in θ with dilation implies that there is no unique M_{ps}.

Going further and following Nova (1982), the data on the η_{max} against D_{min} stress–dilatancy plots (Figs 2 and 3) can be represented by

\[ η_{max} = M - (1 - N)D_{min}^p \]  (10)

because elastic strain rates are zero at peak strength and hence D = D^p. Equation (10) is the general case of equation (2), and equation (10) may be used as a flow rule to derive a family of yield surfaces for CT models (Jeffries, 1993).

For yield surfaces derived from equation (10), N controls the ratio of mean stress at critical to that at isotropic normal compression. Since this ratio of mean stresses cannot be a function of θ under the proposition of a unique CSL (a key assumption of the Cambridge School, and experimentally reasonable, as shown by Been et al., 1991), and since the isotropic NCL is indeed unique, it then follows that N must be independent of θ.

Assuming that N is independent of θ, using N from the triaxial tests allows M to be computed for each plane strain test by rearranging equation (10) as both η_{max} and D_{min} are measured, and M is independent of η by definition. Fig. 5 shows M calculated in this way plotted against θ. What function should be adopted for M(θ)?

Both the Mohr–Coulomb criterion and the Matsuoka–Nakai (Matsuoka & Nakai, 1974) criterion are of interest for representing M(θ). The Mohr–Coulomb criterion with a constant critical friction angle matches commonly held views (e.g. Bolton, 1986; Schanz & Vermeer, 1996). The Matsuoka–Nakai criterion is smooth in the π-plane and is based on the physically appealing concept of spatially mobilised planes (Matsuoka, 1984).

Taking triaxial compression as the reference condition, and hence M_{tc} as the reference soil property, the Mohr–Coulomb criterion for M(θ) is

\[ M = (3\sqrt{3})/[\cos θ(1 + 6/M_{tc}) - \sqrt{3}\sin θ] \]  (11a)

The Matsuoka–Nakai criterion has M(θ) as implicit:

\[ \frac{27 - 3M^2}{3 - M^2 + \frac{2}{3}M^3}\sin(\theta)\left[\frac{1}{2} - \sin^2(\theta)\right] = \frac{27 - 3M_{tc}^2}{3 - M_{tc}^2 + \frac{2}{3}M_{tc}^3} \]  (11b)

Because there is no analytical solution of equation (11b) for M(θ), it is convenient to use the bisection algorithm to find M for θ of interest such that M_{tc} ≤ M ≤ M_{tc}. Fig. 5 compares the Mohr–Coulomb and Matsuoka–Nakai criteria for Brasted sand M_{tc} = 1.27. Also shown in Fig. 5 as a dotted line is the average of M from the Mohr–Coulomb and Matsuoka–Nakai criteria. The plane strain data cluster around this average M line for the range of Lode angles experienced in Cornforth’s tests.

Note that the derivation of the yield surface by integration of equation (2) or equation (10) under normality remains valid whether the Mohr–Coulomb, Matsuoka–Nakai or an average of them is used to represent M(θ).
PLASTIC STRAIN RATE RATIOS AND INCONSISTENT DILATANCY

Using normality in $\pi$-plane

Assuming normality in the $\pi$-plane, strain rates are given by expanding equation (4). By the chain rule:

$$\frac{\partial F}{\partial \dot{\alpha}} = \frac{\partial F}{\partial \dot{\sigma}_{m_{1}}} + \frac{\partial F}{\partial \dot{\sigma}_{m_{2}}} + \frac{\partial F}{\partial \dot{\sigma}_{m_{3}}} + \frac{\partial F}{\partial \dot{\theta}} \frac{\partial \theta}{\partial \dot{\alpha}}$$

(12)

The leading terms in the pairs of differentials on the right-hand side of equation (12) are partial derivatives of the yield surface with respect to its defining stress invariants. The trailing terms are the derivatives of the invariants themselves. Of these two classes of term, the former depends on the constitutive model and is model specific. The latter terms are standard and are given in Table 1 for convenience.

The material derivatives shown in Table 2 depend on the chosen idealisation for $M$. Derivatives for three alternative idealisations are given: the previously discussed Mohr–Coulomb and Matsuoka–Nakai idealisations and, for comparison, simple Drucker–Prager ($M = \text{constant}$).

Inconsistency now becomes apparent. Evaluating $\dot{\epsilon}_{1}^{p}$, $\dot{\epsilon}_{2}^{p}$, $\dot{\epsilon}_{3}^{p}$ using equation (12) and the partial derivatives in Tables 1 and 2, and introducing the local variables $z_{2} = \dot{\epsilon}_{2}^{p}/\dot{\epsilon}$ and $z_{3} = \dot{\epsilon}_{3}^{p}/\dot{\epsilon}$, dilatancy follows from equations (7) and (8) as

$$D^{p} = \frac{3(1 + z_{2} + z_{3})}{(\sin \theta + \sqrt{3} \cos \theta) - 2 \sin(\theta) z_{2} + (\sin \theta - \sqrt{3} \cos \theta) z_{3}}$$

(13)

Figure 6 illustrates $D^{p}$ from normality for both typical contractive ($M_{c} = 1.27$, $N = 0$, $\eta = 0.77$) and dilative ($M_{c} = 1.27$, $N = 0$, $\eta = 1.77$) conditions for Mohr–Coulomb and Matsuoka–Nakai idealisations of $M$.

Also shown in Fig. 6 is the dilatancy from the stress–dilatancy relationship (equation (10)), and which varies with $\theta$ inversely as $M$ because the ratio $(1 - \eta/M)$ is constant for any yield surface hardness (there is only a single value of $\sigma_{m_{1}}$ at any instant). Thus

Table 1. Summary of stress partial differentials (for $\sigma_{m}$, others by cyclic rotation)

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial \sigma_{m}}{\partial \dot{\alpha}}$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{\partial \sigma_{m}}{\partial \dot{\sigma}<em>{m</em>{1}}}$</td>
<td>$\frac{2 \sigma_{m} - \sigma_{m_{1}} - \sigma_{m_{2}}}{2 \sigma_{m}}$</td>
</tr>
<tr>
<td>$\frac{\partial \theta}{\partial \dot{\sigma}<em>{m</em>{1}}}$</td>
<td>$\frac{2 \dot{\epsilon}<em>{1} \dot{\epsilon}</em>{3} + \dot{\epsilon}<em>{1} \dot{\epsilon}</em>{2} + \dot{\epsilon}<em>{2} \dot{\epsilon}</em>{3}}{3 \cos(\theta) \sigma_{m}}$</td>
</tr>
</tbody>
</table>

Table 2. Summary of material partial differentials for alternative $M$ idealisations

<table>
<thead>
<tr>
<th>Term</th>
<th>Drucker–Prager</th>
<th>Mohr–Coulomb</th>
<th>Matsuoka–Nakai</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial F}{\partial \sigma_{m_{1}}}$</td>
<td>$M_{c}^{2} \cos \theta(1 - 4 \sin^{2} \theta)$</td>
<td>$M_{c}^{2} \cos \theta(1 - 4 \sin^{2} \theta)$</td>
<td>$M_{c}^{2} \cos \theta(1 - 4 \sin^{2} \theta)$</td>
</tr>
<tr>
<td>$\frac{\partial F}{\partial \sigma_{m_{2}}}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{\partial F}{\partial \sigma_{m_{3}}}$</td>
<td>0</td>
<td>$\frac{\dot{\sigma}<em>{m</em>{1}}}{M}$</td>
<td>$\frac{\dot{\sigma}<em>{m</em>{1}}}{M}$</td>
</tr>
<tr>
<td>$\frac{\partial F}{\partial M}$</td>
<td>$\frac{\partial \theta}{\partial M}$</td>
<td>$\frac{\partial \theta}{\partial M}$</td>
<td>$\frac{\partial \theta}{\partial M}$</td>
</tr>
<tr>
<td>$\frac{\partial F}{\partial M}$</td>
<td>$\frac{\partial M}{\partial \theta}$</td>
<td>$\frac{\partial M}{\partial \theta}$</td>
<td>$\frac{\partial M}{\partial \theta}$</td>
</tr>
</tbody>
</table>

Dilatancy discrepancies are greatest as the soil changes from triaxial conditions (which often approximate the geostatic state in the ground before engineering works are constructed) through to plane strain (which is what the constructed works commonly produce). Further, some of the implied dilatancies are numerically extreme under non-triaxial conditions, despite being entirely reasonable under triaxial calibration conditions. The situation is actually worse than inconsistency in $D^{p}$. Fig. 7 shows the trends in strain rate ratios corresponding to the trends in dilatancy presented in Fig. 6. The strain rates change sense in moving from triaxial to plane strain conditions, for both Mohr–Coulomb and Matsuoka–Nakai idealisations of $M$. Not only is this physically implausible, it is patently going to be difficult to find any stable numerical procedure for finite element analysis with such a pathological idealisation of soil.

It also follows that standard implementations of CT models based on triaxially calibrated parameters are going to be unrepresentative of ground behaviour commonly encountered in the construction of civil engineering works if $M$ is a function of $\theta$ because plane strain is a dominant design case. But this is only a problem of implementation, and is not something intrinsic to CT models, as we shall now show.

Interpolation for self-consistent dilatation

CT models require normality between the work conjugate stress and strain invariants $\sigma_{m_{1}}$, $\sigma_{m_{2}}$, $\epsilon_{1}$, $\gamma_{4}$ only in going from

$$D^{p} = \frac{M_{c}^{2} \cos \theta(1 - 4 \sin^{2} \theta)}{3 - M_{c} \sin(3 - 4 \sin^{2} \theta) - \frac{9}{A}}$$

where

$$A = \frac{27 - 3 M_{c}^{2}}{3 - M_{c} + \sqrt{3} M_{c}^{2}}$$

and

$$\frac{\partial F}{\partial M} = -\frac{\sigma_{m_{1}}}{M}$$
the flow rule (equation (2)) to the yield surface equation (3), and there is no reason to insist on more than this when going in the reverse direction. However, because there is no role for $\theta$ other than through $\gamma_3$, this leads to an ambiguity in the $\pi$-plane in general for CT models. There is no ambiguity for triaxial conditions, as the symmetry of the triaxial situation removes a degree of freedom, and the flow rule alone is then sufficient to specify the strain rate ratio.

For triaxial extension and compression, from symmetry (and for isotropic elasticity):

**Triaxial compression:**

$$\dot{e}_2 = \dot{e}_3 \Rightarrow \dot{e}_2^p = \dot{e}_3^p \Rightarrow z_{1,tc} = \frac{2D_p - 3}{6 + 2D_p}$$  \hspace{1cm} (15a)

**Triaxial extension:**

$$\dot{e}_2 = \dot{e}_1 \Rightarrow \dot{e}_2^p = \dot{e}_1^p \Rightarrow z_{1,te} = \frac{2D_p - 3}{6 + 2D_p}$$  \hspace{1cm} (15b)

In equations (15a) and (15b), recall that $D_p$ varies with $\theta$ through equation (14).

The first-order idealisation for the variation of strain rate ratios in the $\pi$-plane is a linear interpolation between the two limits given by equations (15a) and (15b) to determine $z_3$ at other Lode angles. However, such an interpolation gives $\theta \approx 9^\circ$ at the critical state in plane strain compared with data that $\theta \approx 15^\circ$ (Fig. 4). A preferable trigonometric interpolation is

$$z_3 = z_{3,tc} - (z_{3,tc} - z_{3,te}) \cos \left(\frac{3\theta + 90}{2}\right)$$  \hspace{1cm} (16)

(for $\theta$ in degrees). Substituting equation (16) in equation (13) and using equation (2) recovers $\dot{e}_3/\dot{e}_1$ consistently with the work dissipation model. Fig. 8 compares this approach with the strain rate ratios from a Drucker–Prager potential surface: a similar well-behaved smoothness is evident.

Because interpolation provides the strain rate ratios directly given $D_p$ and $\theta$ so that differentiation of the potential surface is no longer required, an average of the Mohr–Coulomb and Matsuoka–Nakai criteria can readily be adopted as the best idealisation of $M$ (see Fig. 5). Strain rate vectors and the yield surface in the $\pi$-plane from this interpolation and best-fit model for $M$ are illustrated in Fig. 9. Interestingly, the strain rate vectors are not very different from normality.

### 3-D NORSAND

**Overview**

The above framework for consistent treatment of strain rates in CT models is readily implemented, and is illustrated using NorSand (Jefferies, 1993) as an example. It is perhaps helpful to start by briefly describing NorSand.

NorSand differs from other CT models in that the yield surface does not in general intersect the CSL, although the yield surface has a familiar shape. In NorSand the yield surface is offset by the state parameter $\psi$ and the requirement imposed that $\psi \Rightarrow 0$ as $\gamma_q \Rightarrow \infty \forall \eta > M$, forcing the
Fig. 7. Strain rate ratios from normality in \( \pi \)-plane (\( M_c = 1.27 \) all cases): (a) Mohr–Coulomb criterion, contractive \( D^p_{\text{tc}} = 0.5 \); (b) Matsuoka–Nakai criterion, contractive \( D^p_{\text{tc}} = 0.5 \); (c) Mohr–Coulomb criterion, dilative \( D^p_{\text{tc}} = -0.5 \); (d) Matsuoka–Nakai criterion, dilative \( D^p_{\text{tc}} = -0.5 \) (normality_r5.xls)

Fig. 8. Strain rate ratios from interpolation (normality_r4.xls)
yield surface to move to intersect the CSL as plastic shear strain accumulates. Because the yield surface has been decoupled from the CSL, an additional plastic hardening parameter, $H$, is necessary as $\bar{\kappa}/C_0$ no longer serves as a plastic compliance. Appropriate choices of $\lambda$, $H$ and elastic parameters make Cam-clay (and Granta-gravel) a special case of NorSand.

NorSand, like earlier CT models, was derived for triaxial conditions, and may be extended to general stress states by just five steps:

(a) Stress and strain measures $\bar{\sigma}_m$, $\bar{\sigma}_q$, $\dot{\varepsilon}_m$, $\dot{\gamma}_q$ are substituted for the triaxial measures $p$, $q$, $\dot{v}$, $\dot{\gamma}$ respectively.

(b) Plastic strain rates are made consistent with equation (2) using equations (8), (15) and (16).

(c) $M$ is made a function of $\theta$ and $\psi$.

(d) Volumetric strain hardening is made independent of $\theta$.

(e) Maximum hardening is taken as independent of $\theta$.

The first two steps represent the previous parts of the paper, and it only remains to comment on the last three.

So far, $M$ has been treated as only a function of $\theta$. This follows the standard view in the experimentally orientated literature, which regards $\phi_c$ as a soil property rather than a behaviour. However, an evolving $\phi_c$ is needed to fit the behaviour of soils. In this regard, Dafalias and co-workers (Manzari & Dafalias, 1997; Li et al., 1999) have suggested that $M$ is a function of $\psi$.

The experimental relationship between $D_{\text{min}}$ and $\psi$ is shown in Fig. 10 for some 270 tests on 19 different sands, and can be expressed using a material property $\chi$:

$$D_{\text{min}} = \chi \psi_i$$  \hspace{1cm} (17)

The concept of an image state has been introduced in equation (17) so that there is only a single value of state parameter for any yield surface. The image state is $\psi_i = e - e_{c,i}$, where $e_{c,i}$ is the void ratio of the critical state under image conditions. The image condition is defined in Fig. 11 and further discussed in Jefferies (1993).

At peak strength $D_p = D$, and substitution of equation (17) into equation (10) then allows the original Cam-clay flow rule (equation (2)) to fit the test data with

$$M = M_0 + N \chi \psi_i$$  \hspace{1cm} (18)

where $M_0$ corresponds to the critical friction ratio at $\psi_i = 0$ (the critical state). Interestingly, the product $N \chi$ experimentally approximates unity and as such eliminates $N$ from further consideration. That is:

$$M = M_0 + \psi_i$$  \hspace{1cm} (19)
DILATANCY IN GENERAL CAMBRIDGE-TYPE MODELS

Equation (19) follows Dafalias and co-workers, although it has been derived differently by directly following trends in test data.

Turning to hardening, a fundamental premise of the Cambridge School is that soil tends to the critical state with shear strain; this premise is implicit in CT models that have the yield surface always intersecting the CSL, but must be made explicit in NorSand. A hardening rule complying with this premise is

$$\frac{\dot{\sigma}_{m,i}}{\sigma_m} = H \frac{M}{M_{nc}} \exp \left(1 - \frac{\eta}{M_{nc}}\right) \left[\frac{\sigma_{m,i}}{\sigma_m} \max - \frac{\sigma_{m,c}}{\sigma_m}\right] \theta (20)$$

The property \( H \) in equation (20) is a dimensionless plastic modulus, and might be a function of \( \psi \), but not of either density or mean stress alone, from the requirement of similar behaviour at similar deviation from the critical state.

Self-consistency requires independence of volumetric hardening from \( \theta \) on the NCL, as the behaviour during isotropic compression is unique and cannot be a function of the direction by which the isotropic state is approached. This then requires \( \dot{\sigma}_{m,i}/\dot{\varepsilon}_{m,i} \neq f(\theta) \). Further, \( \theta \) itself becomes indeterminate under isotropic stress but the hardening law must accommodate isotropic compression. These two facts are the reasons why the term \( M/M_{nc} \) is introduced in equation (20). Using equation (2), as \( \eta \to 0 \Rightarrow D' \to M \), from which it follows that \( M_{nc} \dot{\varepsilon}_m = \dot{\varepsilon}_m = \dot{\varepsilon}_m \) eliminating an indefinite condition in the hardening law under isotropic conditions. In passing it can be noted that a volumetric hardening law cannot be used in place of equation (20) if a model is to make the transition from contractive to dilatant behaviour, which is of course the general requirement for dense soils.

The expression \( (\sigma_{m,i}/\sigma_m) \max \) in equation (20) limits the hardening, in effect invoking what is often referred to as a Hvorslev surface. There is no particular reason why this could not be a function of \( \theta \), but there is no loss of generality in taking it as invariant, and doing so avoids introducing additional properties. Accordingly, and again taking triaxial conditions as reference:

$$\frac{\sigma_{m,i}}{\sigma_m} \max = \exp(-\chi_c \psi_i/M_{nc}) \quad (21)$$

Equation (20) represents isotropic softening as well as hardening. Once the soil has reached its limiting dilatancy, the yield surface can contract, if required, by the stress path until the soil reaches the critical state. This behaviour arises because \( (\sigma_{m,i}/\sigma_m) \max \) varies with the current image state parameter through equation (21).

Elastic strains in sand are anisotropic, depend on mean stress, and also depend on void ratio. Several workers have studied the relationship between these factors over several decades, but there is no consensus on the relationship, nor is it clear that any one relationship is generally applicable (e.g. Pestana & Whittle, 1995; Bellotti et al., 1996; Jefferies & Been, 2000). A generalised isotropic Hooke’s law with power law dependence on stress and constant Poisson’s ratio is sufficient here, as the details of elasticity are not crucial to understanding the nature of a critical-state representation of sand for drained loading. The shear modulus, \( G \), and bulk modulus, \( K \), are related by the familiar expressions

$$G = \frac{E}{2(1+\nu)} = \frac{3(1-2\nu)K}{2(1+\nu)} \quad (22)$$

where \( E \) is Young’s modulus and \( \nu \) Poisson’s ratio. Poisson’s ratio is taken as constant, and stress dependence is introduced through the usual relationship:

$$\frac{G}{\sigma_{ref}} = G_{ref} \left( \frac{\sigma_m}{\sigma_{ref}} \right)^g \quad (23)$$

where \( \sigma_{ref} \) is a reference stress, commonly (and arbitrarily— it should be related to grain crushing) taken as 100 kPa.

NorSand is a sparse model, requiring just eight parameters to simulate soil behaviour over the range of accessible void ratios. Of these eight, two define the reference CSL, three define the plastic behaviour, and a further three define the elastic behaviour. Table 3 summarises the parameters and their commonly encountered ranges.

**Computing stress–strain behaviour**

NorSand has no closed-form solution, even for standard laboratory tests where the stress path is known. However, the NorSand equations can be integrated numerically (within a spreadsheet, for example) using the consistency condition along a stress path, allowing convenient comparison of theory and data.

The consistency condition follows from the equations of the yield surface in the usual manner, and is

$$\frac{\eta}{M} = \frac{\dot{\sigma}_{m,i} - \dot{\sigma}_m}{\sigma_{m,i} - \sigma_m} \quad (24)$$

Integration proceeds by imposing a plastic shear strain increment and computing the individual strain increments and the yield surface hardening. This immediately gives the first term within the brackets of equation (24). To proceed, the relationship between \( \eta \) and \( \dot{\sigma}_m \) is required. In the case of drained triaxial tests, the ratio \( \dot{\sigma}_m/\sigma_{m,c} \) is known from the experimental conditions (e.g. \( \dot{\sigma}_m/\sigma_{m,c} = 3 \) for a standard triaxial compression test). In the case of Cornforth’s plane strain experiments discussed here, \( \dot{\sigma}_m = 0 \) and \( \dot{\sigma}_m = \dot{\varepsilon}_m E. \) This is sufficient for direct calculation of the stresses. NorSand is also readily implemented for general stress paths in finite element programs (Shuttle & Jefferies, 1998).

**VALIDATION**

Experimental validation requires duplicate pairs of plane strain and triaxial tests over a range of pressures and densities. The seminal study by Cornforth (1961, 1964)
some 40 years ago provides such data. Cornforth’s data have the attraction of providing drained grouped tests on sand at three densities and two stress levels for each of triaxial compression, triaxial extension and several stress paths in plane strain. This is exactly what is needed to evaluate a CT approach, since CT models claim independence of material properties from density.

The triaxial equipment was the Imperial College apparatus described in Bishop & Henkel (1957). The plane strain apparatus was developed by Wood (1958) at Imperial College under the direction of Professor Bishop and was, in effect, a variation on the triaxial test. The sample was rectangular, 406 mm long by 102 mm high and 52 mm wide, preserving the height : width ratio of the triaxial test. Plane strain was enforced by end platens, with the intermediate principal stress being measured by the infinitely stiff null method. Deviator load was applied vertically by two loading rams using zero axial friction rotating bushings, bearing on a rigid platen at the quarter-length points. Cell pressure was applied in the same way as a triaxial test. Lubricated membranes were used. Although this plane strain apparatus did not have the convenience of modern transducers and data acquisition systems (and their large number of data points), it did provide accurate results for slow drained tests. Fig. 12 illustrates a failed sample after testing in the equipment.

Cornforth’s testing encompassed a wide range of initial void ratios and a somewhat more restricted range of initial stress. The data have been summarised earlier (Fig. 4), but interest here centres on the comparative constitutive behaviour in void ratio and stress level paired triaxial compression and plane strain tests. Table 4 summarises the initial conditions and Fig. 13 gives the stress–strain/dilatancy behaviour measured in three such pairs ranging from loose to dense. As can be seen from Fig. 13, Brasted sand in plane strain is consistently stiffer in plane strain than triaxial compression conditions as well as stronger.

Table 4. Paired tests on Brasted sand (data from Cornforth, 1961)

<table>
<thead>
<tr>
<th>Reference</th>
<th>Test</th>
<th>$e_0$</th>
<th>$\psi_0$</th>
<th>$K_0$</th>
<th>$\sigma_{n,0}$ : kPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>A10–31</td>
<td>Triaxial: C21</td>
<td>0.754</td>
<td>−0.02</td>
<td>0.447</td>
<td>390</td>
</tr>
<tr>
<td>A10–14</td>
<td>Plane strain: P20</td>
<td>0.721</td>
<td>−0.05</td>
<td>0.444</td>
<td>391</td>
</tr>
<tr>
<td>Compact</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A10–34</td>
<td>Triaxial: C25</td>
<td>0.664</td>
<td>−0.11</td>
<td>0.448</td>
<td>389</td>
</tr>
<tr>
<td>A10–12</td>
<td>Plane strain: P18</td>
<td>0.650</td>
<td>−0.12</td>
<td>0.435</td>
<td>395</td>
</tr>
<tr>
<td>Dense</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A10–41</td>
<td>Triaxial: C34</td>
<td>0.570</td>
<td>−0.20</td>
<td>0.379</td>
<td>426</td>
</tr>
<tr>
<td>A10–17</td>
<td>Plane strain: P23</td>
<td>0.572</td>
<td>−0.20</td>
<td>0.381</td>
<td>425</td>
</tr>
</tbody>
</table>

* The reference is to the figure number in Cornforth (1961).
Calibration to triaxial compression data

The CSL is best determined through several undrained triaxial compression tests on loose to compact samples (Been et al., 1991), but no such tests are available for Brasted sand. The CSL parameters were therefore estimated as follows.

Plotting the maximum dilatancy against the initial void ratio for the triaxial compression tests at an initial confining stress of 276 kPa (40 lbf/in²) showed that zero dilatancy corresponded to \( c_v \approx 0.77 \) at this stress. The data at an initial confining stress of 414 kPa (60 lbf/in²) are more scattered and do not allow sensible determination of the critical void ratio for that stress. Therefore \( \lambda = 0.02 \) was adopted as not unreasonable (based on other sands), then giving \( \lambda = 0.902 \) from \( c_v \approx 0.77 \) at 276 kPa.

Elasticity of Brasted sand is uncertain because there are no load–unload stages in the test data to identify the elastic component, nor were modern techniques such as bender elements used. For calibration \( G_{ref} = 500 \) and \( v = 0.2 \) were adopted as not unreasonable for sand at the density and confining stress of Cornforth’s experiments. Elasticity was taken as unaffected by density. The power law exponent was taken as unaffected by density. The power law exponent was taken as unaffected by density.

The critical stress ratio, \( M_{tc} \), was defined by the stress dilatancy plot (Fig. 4), giving \( M_{tc} = 1.27 \) as previously discussed. The dilatancy parameter \( \chi \) was determined by plotting \( D_{dmax} \) in triaxial compression against the initial void ratio, \( \psi_i \) (Fig. 14). The remaining plastic parameter is the hardening modulus, \( H_{tc} \), which was determined through iterative forward modelling of the triaxial compression data.

Iterative forward modelling involves choosing a parameter set, computing the stress–strain behaviour corresponding to that parameter set, and then comparing computed and measured behaviour. Based on the comparison, revised parameters are chosen, with further iterations computed and compared. Iteration optimises the model to the overall data, and Fig. 15 shows the achieved fits for the three triaxial tests of Table 4. In achieving these fits, it turns out that \( H_{tc} \) varies with the state parameter: for the loose test, \( H_{tc} = 75 \); for the compact test, \( H_{tc} = 150 \); for the dense test, \( H_{tc} = 275 \). A dependence of \( H_{tc} \) on \( \psi_i \) is admissible, and for Brasted sand these fits give

\[
H_{tc} = 50 - 1125\psi_i \tag{25}
\]

Performance in plane strain

The comparison of measured and predicted constitutive behaviour in plane strain is presented in Fig. 16. These plane strain simulations used the sand properties obtained in triaxial calibration, described above, without any modification. The three tests plotted in Fig. 14 are the plane strain tests paired with the triaxial tests (see Table 4 and Fig. 11). The evolution of dilatancy, maximum dilatancy, peak strength and the increased stiffness in plane strain are well predicted in all three plane strain tests using the triaxial calibration. The intermediate principal stress is also well predicted in two tests, but diverges a little from the data for the dense sample.

Post-peak behaviour shows a most interesting divergence of theory from data in plane strain. The theory shows a relatively slow reduction in peak strength with axial strain, much as under triaxial conditions. But the experimental results show rapid falls to what appears to be the relevant critical-state condition. It is thought that this is a consequence of strain localisation in the experiments that is not captured in the present numerical simulations, which are based on uniform strain.
CONCLUSION
Cambridge-type (CT) models for soil are an idealised theoretical framework developed from particular postulates, and this gives CT models their predictive power. But inconsistency arises when generalising CT models from triaxial conditions because the postulated plastic working is independent of the Lode angle—which then creates a freedom under general strain paths. The usual approach to this freedom of taking normality in the $/C240/C240$-plane leads to over-specification of the strain rate ratios and, in general, large violations of the postulated work dissipation idealisation.

CT models may be consistently generalised to arbitrary strain paths by requiring the primacy of the work dissipation postulate. The unresolved freedom for the strain rates is handled by interpolation between the limit conditions (triaxial compression and extension) that are fully defined because of symmetry.

Implementation of the approach has been illustrated using the NorSand CT model. Using previously published density and stress level paired tests in triaxial compression and plane strain for loose to dense Brasted sand, NorSand calibrates well to triaxial conditions and then closely predicts the range of behaviour encountered with sands (Jefferies, 1993).

Fig. 14. Peak dilatancy of Brasted sand in triaxial compression against state (dilat_7.xls)

Fig. 15. NorSand calibration to Brasted sand data in triaxial compression: (a) loose, $/psi_0=-0.02$, $H=75$; (b) compact, $/psi_0=-0.11$, $H=150$; (c) dense, $/psi_0=-0.20$, $H=275$ (NorSand_Cornforth_r3.xls)
behaviour measured in plane strain tests. The proposed generalisation is simple, computable, and captures sand
behaviour reasonably well.

The derivations and results presented here are for pre-peak
strength conditions. Further work is required to develop
the details of localisation within a critical-state framework, but
the success of the state parameter in capturing trends in peak
strength and its role as an index to the final state suggest
that this will be fruitful.

It also remains to extend the CT framework to include
principal stress rotation.

ACKNOWLEDGEMENTS

We thank Professor Frans Molenkamp and the Géotechni-
que reviewers for their constructive remarks about the
work presented here. The paper was improved as a result of
their efforts.

NOTATION

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>dilatancy</td>
</tr>
<tr>
<td>e</td>
<td>void ratio</td>
</tr>
<tr>
<td>E</td>
<td>Young’s elastic modulus</td>
</tr>
<tr>
<td>g</td>
<td>exponent for elastic shear modulus; a model parameter</td>
</tr>
<tr>
<td>G</td>
<td>scaling constant for elastic shear modulus; a model parameter</td>
</tr>
<tr>
<td>H</td>
<td>plastic hardening modulus; a model parameter</td>
</tr>
<tr>
<td>K</td>
<td>elastic bulk modulus</td>
</tr>
<tr>
<td>K0</td>
<td>geostatic stress ratio (σ horizontal/σ vertical)</td>
</tr>
<tr>
<td>J2</td>
<td>stress invariant; J2 = σ / σ</td>
</tr>
<tr>
<td>J3</td>
<td>stress invariant; J3 = σ / σ</td>
</tr>
<tr>
<td>M</td>
<td>critical friction ratio; a model parameter</td>
</tr>
<tr>
<td>N</td>
<td>volumetric coupling parameter in stress dilatancy</td>
</tr>
<tr>
<td>p</td>
<td>mean stress (σ = σ mean); overbar denotes effective</td>
</tr>
<tr>
<td>q</td>
<td>triaxial deviator stress; q = σ  − σ</td>
</tr>
<tr>
<td>r</td>
<td>spacing ratio, the ratio of mean normal compression to mean critical-state stress</td>
</tr>
<tr>
<td>s1,2,3</td>
<td>(s2,3 = (σ3 − σ2 − σ3)/3 and others by cyclic rotation)</td>
</tr>
<tr>
<td>ν</td>
<td>elastic Poisson’s ratio; a constant and model parameter</td>
</tr>
<tr>
<td>χ</td>
<td>dilatancy rate parameter (see equation (17)); a model parameter</td>
</tr>
<tr>
<td>ε1,2,3</td>
<td>principal strains (assumed coaxial with principal stresses)</td>
</tr>
</tbody>
</table>

Dilatancy in General Cambridge-Type Models

Fig. 16. NorSand predictions of Brasted Sand behaviour in plane strain: (a) loose, ψ0 = −0.05, H = 85; (b) compact, ψ0 = −0.12, H = 150; (c) dense, ψ0 = −0.20, H = 275 (NorSand_Cornforth_r3.xls)

REFERENCES
